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THERMODYNAMIC CONSTRAINTS FOR CONSTITUTIVE EQUATIONS IN THERMOVISCOELASTICITY: NEW RELATIONSHIPS BETWEEN CROSS EFFECTS†

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For the modified Chen–Gurtin model of a thermoviscoelastic material the consequences of the second law of thermodynamics imposing constraints on the full set of relaxation functions describing both the main and the cross effects are obtained. In the one-dimensional case the functions form a (3×3) matrix and the constraints can be reduced to the condition that the (6×6) matrix constructed from the Fourier transforms of the symmetric and antisymmetric parts of this matrix should be non-negative. The constraints established imply that the instantaneous response matrix must be symmetric. The resulting constraints for the relaxation functions are compared with those already known.

Based on the phenomenological non-equilibrium thermodynamics of complex systems [1–7], an effective method of studying the properties of relaxation functions for materials with a memory, which follow from the second law of thermodynamics, was proposed in [8, 9]. This method was generalized in [10]‡ to the case of the complete system of relaxation functions including functions describing both the main and the cross effects. Below we obtain additional thermodynamic constraints for the relaxation functions which stipulate new relationships between the cross effects and containing the aforesaid properties as a special case.

A modification put forward by the present authors for the model of thermoviscoelastic media developed by Chen and Gurtin in their general thermodynamical theory of materials with memory including second-sound effects [11] is investigated. This interesting model is undoubtedly very promising from the viewpoint of the most adequate description of actual materials. The modification of the model under consideration, involving a special choice of independent variables and the introduction of an appropriate thermodynamic potential, is used because it enables one to construct a thermodynamically consistent linear theory [12] and also because it gives a more compact representation of results. The one-dimensional model is considered for simplicity.

1. NOTATION AND THE BASIC ASSERTIONS OF THE MODIFIED CHEN–GURTIN THEORY

Let $\mathbf{R}$ be the set of real numbers and let $\mathbf{R}^+$ be the set of positive real numbers. For any function $g\colon \mathbf{R}^+ \to \mathbf{R}$ we introduce the summary function $\bar{g}\colon \mathbf{R}^+ \to \mathbf{R}$ by

$$\bar{g}(s) = \int_0^s g(\lambda)d\lambda \tag{1.1}$$

For every time-dependent function $f\colon \mathbf{R} \to \mathbf{R}^+$ and any fixed instant t the history $f^t\colon \mathbf{R}^+ \to \mathbf{R}$ of f up to time t is defined as follows:

$$f^t(s) = f(t-s) \tag{1.2}$$

and the summary history $\bar{f}^t\colon \mathbf{R}^+ \to \mathbf{R}$ of f up to t is defined according to (1.1). We denote by H the Hilbert space of measurable functions $f\colon \mathbf{R}^+ \to \mathbf{R}$ with finite norm $\|\cdot\|$ defined by

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‡See also KOLPASHCHIKOV V. L. and SCHNIPP A. I., Thermodynamics and models of non-classical media. Preprint No. 9, Inst. Teplomassoobmena Akad. Nauk BSSR, Minsk, 1986.

$$\|f\| = \int_0^\infty |f(s)|^2 h(s)ds \tag{1.3}$$

where h: $\mathbf{R}^+ \to \mathbf{R}$ is a continuous decreasing influence function such that

$$\int_0^\infty h(s)sds < \infty$$

We set $\tilde{\mathbf{H}} = \{\tilde{f} \in \mathbf{H} \mid f \in \mathbf{H}\}$ and $\mathbf{H}^+ = \{f \in \mathbf{H} \mid f > 0\}$. For the one-dimensional model under consideration, the triple $\{F(t) > 0, \vartheta(t) > 0, G(t)\}$ of time-dependent functions defined at some point of the medium, F being the deformation gradient, $\vartheta = 1/T$ being the inverse absolute temperature and $G = \partial\vartheta/\partial x$ its gradient, will be called an admissible process at that point if F and ϑ are continuous and piecewise smooth, G is piecewise continuous, and all the three functions are bounded. Each admissible process defines an admissible state

$$\Lambda^t = \{F(t),\ \vartheta(t),\ F^t,\ \vartheta^t,\ \overline{G^t}\} \tag{1.4}$$

of the material under consideration for any fixed time t, where, as can be verified, $F(t) \in R^+$, $\vartheta(t) \in R^+$, $F^t \in H^+$, $\vartheta^t \in \mathbf{H}^+$, $\bar{G}^t \in \tilde{\mathbf{H}}$. The corresponding set $S = \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{H}^+ \times \tilde{\mathbf{H}}$ will be called the state space.

In addition to the laws of conservation of momentum and energy, the following constitutive equations, given by the four state functions (functionals) at every point of the material, are necessary for the complete description of the material

$$S(t) = \hat{S}(\Lambda^t),\quad e(t) = \hat{e}(\Lambda^t),\quad q(t) = \hat{q}(\Lambda^t),\quad \eta(t) = \hat{\eta}(\Lambda^t) \tag{1.5}$$

where S is the Piola–Kirchhoff stress, e is the specific internal energy, q is the heat flow, and η is the specific entropy. The circumflex accent denotes the corresponding real constitutive state functionals. It will be convenient below to introduce the following thermodynamic potential

$$\Phi = e\vartheta - \eta \tag{1.6}$$

for which a constitutive equation

$$\Phi(t) = \hat{\Phi}(\Lambda^t) \tag{1.7}$$

of the same type can be established using (1.5) and (1.6). The set

$$\{F(t), \vartheta(t), G(t);\ S(t), e(t), q(t), \Phi(t)\}$$

of seven time-dependent functions defined at some point, the first three of which represent an admissible process, while the remaining ones are determined by that admissible process according to (1.5) and (1.7), will be called a thermodynamic process at that point.

The functionals in (1.5) and (1.7) are assumed to be continuous and differentiable, and $\hat{\Phi}$ is twice differentiable in S. It follows that the partial derivatives

$$\partial_F \hat{P}(\Lambda^t) = \frac{\partial}{\partial F}\hat{P}(\Lambda^t),\quad \partial_\vartheta \hat{P}(\Lambda^t) = \frac{\partial}{\partial\vartheta}\hat{P}(\Lambda^t) \tag{1.8}$$

exist, and so do the Fréchet partial derivatives

$$\begin{aligned}
\delta_1\hat{P}(\Lambda^t)(f) &= \frac{\partial}{\partial\lambda}\hat{P}(F, \vartheta, F^t + \lambda f, \vartheta^t,\ \overline{G^t})\Big|_{\lambda=0} \\
\delta_2\hat{P}(\Lambda^t)(l) &= \frac{\partial}{\partial\lambda}\hat{P}(F, \vartheta, F^t, \vartheta^t + \lambda l,\ \overline{G^t})\Big|_{\lambda=0} \\
\delta_3\hat{P}(\Lambda^t)(g) &= \frac{\partial}{\partial\lambda}\hat{P}(F, \vartheta, F^t, \vartheta^t, \overline{G^t} + \lambda g)\Big|_{\lambda=0}
\end{aligned} \tag{1.9}$$

where $f, l, g \in \mathbf{H}$, $\delta_m \hat{P}(\Lambda^t)$: $\mathbf{H} \xrightarrow{L} \mathbf{R}$, $m = 1, 2, 3$, $\hat{P}$ being any of the functionals, $\hat{S}, \hat{e}, \hat{q}, \hat{\eta}, \hat{\Phi}$. For $\hat{\Phi}$ there are also the corresponding second derivatives, for example

$$\delta_{ij}^2 \hat{\Phi}(\Lambda^t)(f, g) \tag{1.10}$$

where $f, g, \in \mathbf{H}$, $\delta_{ij}^2 \hat{\Phi}(\Lambda^t)$: $\mathbf{H} \times \mathbf{H} \xrightarrow{L} \mathbf{R}$, $i, j = 1, 2, 3$.

To simplify the notation below, we will introduce the three-dimensional Euclidean space $\mathbf{R}^3$ of elements of the form $a = \{a_1, a_2, a_3\}$, where $a_1, a_2, a_3 \in R$, as well as the Hilbert space $\mathbf{H}^3$ of vector-valued functions Γ: $\mathbf{R}^+ \to \mathbf{R}^3$ of the form $\Gamma = \{f_1, f_2, f_3\}$, where $f_1, f_2, f_3 \in \mathbf{H}$. The scalar product and norm in $\mathbf{R}^3$ and $\mathbf{H}^3$ are defined in the usual way

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad \|\Gamma\| = (\|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2)^{1/2} \tag{1.11}$$

Using this notation, we can introduce the total Fréchet derivative

$$\delta \hat{P}(\Lambda^t)(\Gamma) = \delta \hat{P}(\Lambda^t)(\{f_1, f_2, f_3\}) =$$
$$= \delta_1 \hat{P}(\Delta^t)(f_1) + \delta_2 \hat{P}(\Lambda^t)(f_2) + \delta_3 \hat{P}(\Lambda^t)(f_3),$$
$$\Gamma = \{f_1, f_2, f_3\} \in H_3$$

2. THE SECOND LAW OF THERMODYNAMICS

The first three constitutive equations in (1.5) together with the momentum and energy balance equations form a closed system of equations, which enables us to describe the thermodynamic processes in the materials under consideration, provided that the corresponding boundary conditions are given along with the volume forces and heat sources. Any admissible processes for an arbitrarily chosen point of the medium can be obtained by solving this system, since the volume forces and heat sources are arbitrary. To exclude thermodynamically inadmissible processes the law of thermodynamics should in some way restrict the class of constitutive equations.

In the theory under consideration the second law of thermodynamics is based on the Clausius–Duhem inequality, which expresses the requirement that the internal production of entropy σ is positive, provided that the external mass source of entropy is equal to the mass source of heat r divided by the absolute temperature ($r/T = r\vartheta$), and the entropy flow is equal to the heat flux divided by the absolute temperature $q\vartheta$

$$\sigma \equiv \rho\dot{\eta} + \operatorname{div}(\mathbf{q}\vartheta) - \rho r\vartheta \geqslant 0 \tag{2.1}$$

where ρ is the density of the medium and a dot above a symbol will from now on denote the total derivative with respect to time.

Taking into account the energy balance equation

$$\rho\dot{e} = -\operatorname{div}\mathbf{q} + \rho\mathbf{S} \cdot \dot{\mathbf{F}} + \rho r \tag{2.2}$$

and using (1.6), we can reduce (2.1) in the one-dimensional case to the form

$$-\dot{\Phi} + e\dot{\vartheta} + \vartheta S\dot{F} + \frac{1}{\rho} qG \geqslant 0 \tag{2.3}$$

The second law of thermodynamics can be stated as follows.

Postulate TD. Inequality (2.3) is satisfied for all thermodynamic processes.

Necessary and sufficient conditions for this postulate to be satisfied are given in the following theorem, which is analogous to the corresponding theorem in [11].

Theorem CG. Postulate TD is satisfied if and only if the constitutive equations satisfy the following relationships

$$\hat{S}(\Lambda^t) = \frac{1}{\vartheta(t)} \partial_F \hat{\Phi}(\Lambda^t), \quad \hat{e}(\Lambda^t) = \partial_\vartheta \hat{\Phi}(\Lambda^t) \tag{2.4}$$

$$\hat{q}(\Lambda^t) = \rho \delta_3 \hat{\Phi}(\Lambda^t)(1^+), \quad \delta\hat{\Phi}(\Lambda^t)(\{\dot{\vartheta}^t, \dot{F}^t, -G^t\}) \leq 0$$

Here $1^+ \in \mathbf{H}$ is a constant function equal to unity everywhere in $\mathbf{R}^+$.

We now define an equilibrium state Λ_0 as a state of the following form

$$\Lambda_0 = \{F, \vartheta, F^+, \vartheta^+, 0^+\} \tag{2.5}$$

where F^+, ϑ, 0^+ are constant functions from $\mathbf{H}$ and $\tilde{\mathbf{H}}$ equal, respectively, to F, ϑ and 0 on the whole set $\mathbf{R}^+$.

The aim of the study below is to obtain thermodynamic constraints for the linear parts of the first three constitutive functions in (1.5) appearing in the momentum and energy transfer equations and defined by the first-order Fréchet derivatives. Introducing the relaxation functions using the Riesz lemma on the representation of a linear functional in Hilbert space, we can represent the linear parts of the functionals in question by integrals

$$\delta_m \hat{\psi}_n(\Lambda_0)(f) = \int_0^\infty R'_{mn}(s) f(s) ds, \quad R_{mn}(\infty) = 0; \quad m, n = 1, 2, 3 \tag{2.6}$$

$$\hat{\psi}_1 = \vartheta \hat{S}, \quad \hat{\psi}_2 = \hat{e}, \quad \hat{\psi}_3 = -\hat{q}/\rho$$

The prime denotes differentiation and the operators δ_m are defined in (1.9).

Nine relaxation functions are introduced here. They are analogues of the phenomenological transfer coefficients in classical theory. Each of them is named according to the thermodynamic quantity whose relaxation it describes and the independent variable whose variation is responsible for the relaxation process. For example, R_{11} is the stress versus the deformation relaxation function, R_{32} is the heat flux versus the temperature relaxation function, etc.

3. THERMODYNAMICS AND PROPERTIES OF THE RELAXATION FUNCTIONS

The study of the properties of the relaxation functions below is based on a lemma following from Theorem CG. The proof of this lemma is omitted here. It is based on expanding the left-hand side of the inequality in (2.4) in a functional Taylor series, so it does not differ much from the proofs of similar results in [8, 9].

Lemma. If Postulate TD is satisfied, $\hat{\Phi}$ has the following property: for any equilibrium state Λ_0 and any bounded $\Gamma = \{f_1, f_2, f_3\} \in \mathbf{H}^3$ such that

$$\Gamma' = \left\{\frac{df_1}{ds}, \frac{df_2}{ds}, \frac{df_3}{ds}\right\} \in H$$

the inequality

$$\partial_F \delta\hat{\Phi}(\Lambda_0)(\Gamma') f_1(0) + \partial_\vartheta \delta\hat{\Phi}(\Lambda_0)(\Gamma') f_2(0) -$$

$$-\delta_1 \delta\hat{\Phi}(\Lambda_0)(\Gamma', 1^+) f_3(0) + \delta^2 \hat{\Phi}(\Lambda_0)(\Gamma', \Gamma) \geq 0 \tag{3.1}$$

is satisfied. Since the differential operators ∂ and δ commute, one can use the first three relationships in (2.4) and express $\partial_F\hat{\Phi}$, $\partial_\vartheta\hat{\Phi}$, and $\delta_3\hat{\Phi}$ in (3.1) in terms of $\hat{S}$, $\hat{\eta}$ and $\hat{q}$.

Moreover, if the following vector-valued functional $\hat{\Psi}$ is introduced on S

$$\hat{\Psi}(\Lambda^t) \equiv \{\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3\} = \left\{\vartheta \hat{S}(\Lambda^t), \ \hat{e}(\Lambda^t), \ -\frac{1}{\rho}\hat{q}(\Lambda^t)\right\} \tag{3.2}$$

then (3.1) can be reduced to

$$\delta\hat{\Psi}(\Lambda_0)(\Gamma')\Gamma(0)+\delta^2\hat{\Phi}(\Lambda_0)(\Gamma',\Gamma)\geq 0 \tag{3.3}$$

We consider two specific choices of Γ and $\mathbf{H}^3$ satisfying the hypothesis of the lemma

$$\Gamma_1 = \mathbf{a}C_\omega(s) + \mathbf{b}S_\omega(s) \tag{3.4}$$

$$\Gamma_2 = \mathbf{a}S_\omega(s) + \mathbf{b}C_\omega(s) \tag{3.5}$$

where

$$\mathbf{a} = \{a_1, a_2, a_3\} \in \mathbf{R}^3, \quad \mathbf{b} = \{b_1, b_2, b_3\} \in \mathbf{R}^3$$

$$S_\omega(s) = \sin(\omega s) \in \mathbf{H}, \quad C_\omega(s) = \cos(\omega s) \in \mathbf{H}, \quad \omega \in \mathbf{R}^+$$

Substituting (3.4) and then (3.5) into (3.3), we obtain

$$\delta\hat{\Psi}(\Lambda_0)(-\omega\mathbf{a}S_\omega+\omega\mathbf{b}C_\omega)\mathbf{a}+\delta^2\hat{\Phi}(\Lambda_0)(-\omega\mathbf{a}S_\omega+\omega\mathbf{b}C_\omega,\ \mathbf{a}C_\omega+\mathbf{b}S_\omega)\geq 0 \tag{3.6}$$

$$-\delta\hat{\Psi}(\Lambda_0)(\omega\mathbf{a}C_\omega+\omega\mathbf{b}S_\omega)\cdot\mathbf{b}+\delta^2\hat{\Phi}(\Lambda_0)(\omega\mathbf{a}C_\omega+\omega\mathbf{b}S_\omega,\ \mathbf{a}S_\omega-\mathbf{b}C_\omega)\geq 0 \tag{3.7}$$

Taking into account that the second-order Fréchet derivative for a fixed state Λ_0 is a symmetric bilinear functional on $\mathbf{H}^3\times\mathbf{H}^3$(i.e. such that $\delta^2\hat{\Phi}(\Lambda_0)(\Gamma_1,\Gamma_2)=\delta^2\hat{\Phi}(\Lambda_0)(\Gamma_2,\Gamma_2)$, it can be shown that the terms with $\delta^2\hat{\Phi}$ in (3.6) and (3.7) are equal to one another with opposite signs. Therefore, when these inequalities are added, the terms cancel one another and we get

$$\delta\hat{\Psi}(\Lambda_0)(\omega\mathbf{a}S_\omega-\omega\mathbf{b}S_\omega)\cdot\mathbf{a}+\delta\hat{\Psi}(\Lambda_0)(\omega\mathbf{a}C_\omega-\omega\mathbf{b}S_\omega)\cdot\mathbf{b}\leq 0 \tag{3.8}$$

Representing $\hat{\Psi}$ by the relaxation functions (2.6) and changing to the detailed notation, we can rewrite (3.8) on dividing it by $\omega>0$ in the form

$$\sum_{n=1}^{3}\sum_{m=1}^{3}a_na_m\int_0^\infty R'_{nm}(s)\sin(\omega s)ds-\sum_{n=1}^{3}\sum_{m=1}^{3}a_nb_m\int_0^\infty R'_{nm}(s)\cos(\omega s)ds+$$

$$+\sum_{n=1}^{3}\sum_{m=1}^{3}b_na_m\int_0^\infty R'_{nm}(s)\cos(\omega s)ds+\sum_{n=1}^{3}\sum_{m=1}^{3}b_nb_m\int_0^\infty R'_{nm}(s)\sin(\omega s)ds\leq 0 \tag{3.9}$$

We introduce the antisymmetric and symmetric parts $\| R^-_{nm}\|$ and $\| R^+_{nm}\|$ of $\| R_{nm}\|$

$$R^-_{nm}=\frac{1}{2}(R_{nm}-R_{mn}),\quad R^+_{nm}=\frac{1}{2}(R_{nm}+R_{mn}),\quad n,m=1,2,3 \tag{3.10}$$

and the cosine and sine Fourier transforms of $R(s)$

$$R|_c(\omega)=\int_0^\infty R(s)\cos(\omega s)ds,\quad R|_s(\omega)=\int_0^\infty R(s)\sin(\omega s)ds \tag{3.11}$$

Using this notation, we can rewrite (3.9) as follows:

$$\sum_{n=1}^{3}\sum_{m=1}^{3}R^{+'}_{nm}\Big|_s(\omega)a_na_m+2\sum_{n=1}^{3}\sum_{m=1}^{3}R^{-'}_{nm}\Big|_c(\omega)b_na_m+$$

$$+\sum_{n=1}^{3}\sum_{m=1}^{3}R^{+'}_{nm}\Big|_s(\omega)b_nb_m\leq 0 \tag{3.12}$$

We introduce a new (6×6) matrix $\| r_{nm}\|$

$$r_{nm}(\omega)=-R^{+'}_{nm}\Big|_s(\omega)\ \text{for}\ n,\ m=1,2,3$$

$$r_{nm}(\omega)=-R^{+'}_{n-3,m-3}\Big|_s(\omega)\ \text{for}\ n,\ m=4,5,6 \tag{3.13}$$

$$r_{nm}(\omega) = -R^{-'}_{n,m-3}\Big|_c(\omega) \text{ for } n = 1, 2, 3; \; m = 4, 5, 6$$

$$r_{nm}(\omega) = -R^{-'}_{m,n-3}\Big|_c(\omega) \text{ for } n = 4, 5, 6; \; m = 1, 2, 3$$

This symmetric matrix can be represented in the form of a partitioned matrix

$$\| r_{nm}(\omega) \| = \left\| \begin{array}{c|c} \left\| R^{+'}_{nm}\big|_s(\omega) \right\| & \left\| R^{-'}_{nm}\big|_c(\omega) \right\| \\ \hline \left\| R^{-'}_{mn}\big|_c(\omega) \right\| & \left\| R^{+'}_{nm}\big|_s(\omega) \right\| \end{array} \right\| \tag{3.14}$$

If the six-dimensional vector

$$\tilde{a}_k = \begin{cases} a_k & \text{for } k = 1,2,3 \\ b_{k-3} & \text{for } k = 4,5,6 \end{cases} \tag{3.15}$$

is also introduced, then (3.12) can be rewritten using the above notation as the following quadratic form

$$\sum_{n=1}^{6} \sum_{m=1}^{6} r_{nm}(\omega) \tilde{a}_n \tilde{a}_m \geq 0 \tag{3.16}$$

for any $\tilde{a}_n$ ($n = 1, 2, \ldots, 6$) and $\omega \geq 0$.

This implies that (3.14) must be a non-negative definite matrix. The following theorem has therefore been proved.

Theorem 1. If Postulate TD is satisfied, the matrix $\| r_{nm}(\omega) \|$ constructed from the relaxation functions (2.6) in accordance with (3.10), (3.11) and (3.14) must be non-negative definite.

A necessary and sufficient condition for the matrix to be non-negative definite is that all its principal minors must be non-negative definite. In general, a (6 × 6) matrix has 63 principal minors. However, in the case in question the matrix (3.14) contains many repeated terms, as can be seen from its full form, which can be obtained from (3.14) by integrating by parts those of its elements which contain the sine Fourier transform

$$\left\| \begin{array}{cccccc} \omega R^{+}_{11}\big|_c & \omega R^{+}_{12}\big|_c & \omega R^{+}_{13}\big|_c & 0 & R^{-'}_{12}\big|_c & R^{-'}_{13}\big|_c \\ \omega R^{+}_{12}\big|_c & \omega R^{+}_{22}\big|_c & \omega R^{+}_{23}\big|_c & R^{-'}_{12}\big|_c & 0 & R^{-'}_{23}\big|_c \\ \omega R^{+}_{13}\big|_c & \omega R^{+}_{23}\big|_c & \omega R^{+}_{33}\big|_c & -R^{-'}_{13}\big|_c & -R^{-'}_{23}\big|_c & 0 \\ 0 & -R^{-'}_{12}\big|_c & -R^{-'}_{13}\big|_c & \omega R^{+}_{11}\big|_c & \omega R^{+}_{12}\big|_c & \omega R^{+}_{13}\big|_c \\ R^{-'}_{12}\big|_c & 0 & -R^{-'}_{23}\big|_c & \omega R^{+}_{12}\big|_c & \omega R^{+}_{22}\big|_c & \omega R^{+}_{23}\big|_c \\ R^{-'}_{13}\big|_c & R^{-'}_{23}\big|_c & 0 & \omega R^{+}_{13}\big|_c & \omega R^{+}_{23}\big|_c & \omega R^{+}_{33}\big|_c \end{array} \right\| \tag{3.17}$$

There are therefore just 32 different minors among the principal minors of this matrix. Consequently, the above condition that the matrix must be non-negative definite yields 32 inequalities imposing constraints upon the relaxation function matrix. The first seven of these inequalities representing the minors formed by the elements of the upper left quarter of (3.17) contain only the symmetric part of the relaxation function matrix $\| R_{ij} \|$ and are identical with the inequalities obtained in [10]. The remaining 25 inequalities are new and impose constraints on the antisymmetric part of the matrix also, giving rise to new relationships between the cross effects.

To state an interesting conclusion which follows from these new constraints we introduce the following definition.

We will call $\| R_{ij}(0) \|$ ($i, j = 1, 2, 3$) the instantaneous response matrix. It describes the difference between the response of the system to an instantaneous (jump-like) variation of the independent variables and the response to a quasistationary variation of these variables of the same magnitude.

Conclusion. The instantaneous response matrix is symmetric, i.e.

$$R_{ij}(0) = R_{ji}(0), \quad i, j = 1, 2, 3; \quad i < j \tag{3.18}$$

To prove this conclusion we will write down the inequalities containing the non-negative definiteness conditions for the three second-order principal minors in (3.17) formed by the elements of the rows and columns numbered 1 and 5, 1 and 6, and 2 and 6, respectively, reducing the terms containing the antisymmetric part of $\| R_{ij} \|$ to the sine Fourier transform by integration by parts

$$\omega^2 R_{ii}^+\big|_c(\omega) R_{jj}^+\big|_c(\omega) - \left[R_{ij}^-(0) + \omega R_{ij}^-\big|_s(\omega)\right]^2 \geqslant 0$$

$$i, j = 1, 2, 3; \quad i < j \tag{3.19}$$

Letting ω tend to zero in (3.19), by the integrability of the relaxation functions we get

$$-(R_{ij}^-(0))^2 \geqslant 0, \quad i, j = 1, 2, 3; \quad i < j$$

It follows that

$$R_{ij}^-(0) = 0, \quad i, j = 1, 2, 3; \quad i < j$$

Taking (3.10) into account, this is equivalent to the assertion (3.18) to be proved.

Here an analogy with Onsager's reciprocity relationships can be seen explicitly. However, unlike the usual phenomenological terminology for irreversible processes [13], here these properties follow from the irreversibility principle (the positivity of the production of entropy) and do not require any special postulate. At the same time one should take into account that the physical meaning of the quantities in (3.18) differs from that of Onsager's phenomenological coefficients.

4. COMPARISON WITH KNOWN RESULTS

It is interesting to compare the constraints obtained here for the relaxation functions with other known results, for example, with the properties of the stress versus deformation relaxation function G described in [2, Chapter 6]. To do this, it is convenient to represent (3.16) in a slightly different form. In addition to the real Euclidean space $\mathbf{R}^3$, we introduce the three-dimensional complex Euclidean space $\mathbf{C}^3$ as well as the space $\mathbf{L}(\mathbf{R}^3)$ of linear transformations of $\mathbf{R}^3$ into itself (real matrices) and the space $\mathbf{L}(\mathbf{C}^3)$ of linear transformations of $\mathbf{C}^3$ into itself (complex matrices). Then the relaxation function matrix $\| R_{nm} \|$ can be considered as a function from $\mathbf{R}^+$ to $\mathbf{L}(\mathbf{R}^3)$, which will be denoted by $\mathbb{R}$. We introduce a new matrix-valued function $\tilde{\mathbb{R}}$ defined on the whole set $\mathbf{R}$ in terms of $\mathbb{R}$ as follows:

$$\tilde{\mathbb{R}}(s) = \begin{cases} \mathbb{R}(s), & s \geqslant 0 \\ \mathbb{R}'(-s), & s < 0 \end{cases} \tag{4.1}$$

The Fourier transform of $\tilde{\mathbb{R}}\,|_F : \mathbf{R} \to \mathbf{L}(\mathbf{C}^3)$ can be defined by

$$\tilde{\mathbb{R}}|_F(\omega) = \int_{-\infty}^{\infty} \tilde{\mathbb{R}}(s) e^{-i\omega s} ds \tag{4.2}$$

The Fourier transforms of the derivatives of R_{nm} in (3.12) can be reduced to the Fourier transforms of R_{nm} by integration by parts. Using the matrix notation introduced above and taking (3.10) into account, inequality (3.12) transformed in this way or the equivalent inequality (3.9) can be written in the form

$$\omega \mathbf{a} \cdot \mathbb{R}|_c(\omega)\mathbf{a} + \mathbf{b} \cdot \left(\mathbb{R}|_s(\omega)\omega - \mathbb{R}^T|_s(\omega)\omega + R(0) - \mathbb{R}^T(0)\right) \cdot \mathbf{a} + \omega \mathbf{b} \mathbb{R}|_c(\omega) \cdot \mathbf{b} \geqslant 0 \tag{4.3}$$

Since $\mathbb{R}^T(0) = \mathbb{R}(0)$ by Theorem 1 and $\omega \geqslant 0$ by assumption, the following assertion can easily be deduced.

Assertion 1. The constraints for the relaxation functions stated in Theorem 1 are satisfied if and only if the matrix-valued function $\tilde{\mathbb{R}}$ satisfies the conditions

$$\tilde{\mathbb{R}}(0) = \tilde{\mathbb{R}}^T(0), \quad \mathbf{c}^* \tilde{\mathbb{R}}|_F(\omega)\mathbf{c} \geqslant 0 \tag{4.4}$$

for all $\mathbf{c} \in \mathbf{C}^3$ and $\omega \in \mathbf{R}$ (the asterisk denotes complex conjugation).

It is easy to establish the equivalence of (4.4) and (4.3) setting $\mathbf{c} = \mathbf{a} + i\mathbf{b}$ and separating the real and imaginary parts in the second relationship in (4.4).

At the same time, by Bochner's theorem generalized to the case of matrix-valued functions,† the second relationship in (4.4) is equivalent to the condition

$$\int_0^T \int_0^t \Gamma(t) \cdot \mathbb{R}(t-s)\Gamma(s)dsdt \geqslant 0 \tag{4.5}$$

for any locally square integrable $\Gamma: \mathbf{R}^+ \to \mathbf{R}^3$ and $T > 0$.

This condition is equivalent to the so-called dissipativity condition for $\mathbb{R}(s)$ considered in [2] for $\mathbf{G}(s)$. We remark, however, that the relaxation functions have been introduced here in a slightly different way, since it was assumed that $R_{nm}(\infty) = 0$. For example, the stress versus deformation relaxation function $R_{11}(s)$ introduced here corresponds to the function $\mathbf{G}(s) - \mathbf{G}(\infty)$ in Chapter 6 of [2] and $R_{11}^0 - R_{11}(0)$, where $R_{11}^0 = \partial_F \hat{\psi}_1(\Lambda_0)$, corresponds to $\mathbf{G}(\infty)$. We can also draw a wider analogy by comparing the matrix-valued stress versus deformation relaxation function $\mathbf{G}(s)$ considered in [2] with the matrix-valued relaxation function

$$\overline{\mathbb{R}}(s) = \mathbf{R}^0 + \mathbb{R}(s) \tag{4.6}$$

representing all forms of relaxation. Here

$$\overline{\mathbb{R}}^0 = \begin{Vmatrix} \partial_F \hat{\psi}_1(\Lambda_0) & \partial_F \hat{\psi}_2(\Lambda_0) & \partial_F \hat{\psi}_3(\Lambda_0) \\ \partial_\vartheta \hat{\psi}_1(\Lambda_0) & \partial_\vartheta \hat{\psi}_2(\Lambda_0) & \partial_\vartheta \hat{\psi}_3(\Lambda_0) \\ \delta_3 \hat{\psi}_1(\Lambda_0)(1^+) & \delta_3 \hat{\psi}_2(\Lambda_0)(1^+) & \delta_3 \hat{\psi}_3(\Lambda_0)(1^+) \end{Vmatrix} \tag{4.7}$$

and the functionals $\hat{\psi}_n$ are defined as in (2.6).

Applying the differentiation operators ∂_ϑ, ∂_F, $\delta^3(1^+)$ to the first three relationships in (2.4), it can be shown that

$$\partial_F \hat{\psi}_2(\Lambda_0) = \partial_\vartheta \hat{\psi}_1(\Lambda_0), \quad \partial_F \hat{\psi}_3(\Lambda_0) = \delta_3 \hat{\psi}_1(\Lambda_0)(1^+)$$

$$\partial_\vartheta \hat{\psi}_3(\Lambda_0) = \delta_3 \hat{\psi}_2(\Lambda_0)(1^+)$$

i.e. $\mathbb{R}^0$ is a symmetric matrix.

Then, since the relationships (3.18) are satisfied, the matrix $\overline{\mathbb{R}}(0)$ will also be symmetric

$$\overline{\mathbb{R}}(0) = \mathbf{R}^T(0) \tag{4.8}$$

Taking (4.6) into account, constraints (4.8) and (4.5) are equivalent to the condition of "compatibility with thermodynamics" for $\mathbb{R}(s)$ stated for $\mathbf{G}(s)$ in assertion 2b in [2, Chapter 6]. We note that, using the same argument in [2], one can prove the following properties of the matrix-valued function $\mathbb{R}$ starting from (4.5)

$$\mathbb{R}(0) - \mathbb{R}(s) \geqslant 0$$

and

$$\mathbb{R}(0) \geqslant 0, \quad -\mathbb{R}'(0) \geqslant 0 \tag{4.9}$$

where the inequality signs means that the corresponding matrices are non-negative definite.

The properties of the relaxation functions of type (4.9) have important applications in the study of wave propagation and also in the proof of the theorem on the uniqueness of solutions of the system of linear field equations of combined thermoviscoelasticity.

†This generalization can be found, for example, in KOLPASHCHIKOV V. L. and SCHNIPP A. I., Linear thermodynamical systems with a memory: a necessary and sufficient condition for the existence of a non-equilibrium thermodynamical potential. Preprint No. 37, Inst. Teplomassoobmena Akad. Nauk. BSSR, Minsk, 1987.

Note that, unlike the results in [2], these assertions are related to the complete system of relaxation functions describing all three types of relaxation, including cross effects. Moreover, they are obtained for the non-linear constitutive functionals, the linear parts of which are represented by the relaxation functions.

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